PROBLEMS OF THE CALCULUS OF VARIATIONS INVARIANT UNDER A CONTINUOUS GROUP.

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In order to make clear the purpose of this paper, it will be well to recall the results of a few of the typical problems in the calculus of variations. It is known that in the problem of the minimum surface of revolution, the integral to be minimized is

$$I = \int_{x_1}^{x_2} y\sqrt{1 + y'^2}dx,$$

and the extremals are of the form

$$(1) \qquad y - \alpha = a\varphi\left(\frac{x - b}{a}\right) \qquad (a > 0),$$

where, in this problem, the function φ is the hyperbolic cosine and $\alpha = 0$. In case of the brachistochrone problem, the integral is

$$I = \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2}}{\sqrt{y - \alpha}} dx$$

and has for its extremals the cycloids

$$x = a(\theta - \sin \theta) + b, \qquad y = a(1 - \cos \theta) + \alpha,$$

which, if θ is eliminated, have the form (1). For Newton's problem of the surface of minimum resistance,

$$I = \int_{x_1}^{x_2} \frac{yy'^3}{1 + y'^2} dx$$

and the extremals are

$$x - b = a\left(\log p + \frac{1}{p^2} + \frac{3}{4p^4}\right), \qquad y = a\left(\frac{1}{p^3} + \frac{2}{p} + 1\right),$$

which can also be put in the form (1) by the elimination of p.

In these problems, and others, the equations of the extremals all come under the type (1) and the curves represented by (1) all have the property of being invariant under the two-parameter group of stretchings and

translations

$$x' = sx + t,\quad y' - \alpha = s(y - \alpha) \qquad (\alpha = \text{const.}), \tag{2}$$

a fact that can be easily verified.

It is proposed in this paper, first, to find the types of integrals

$$I = \int_{x_1}^{x_2} f(x, y, y')dx, \tag{3}$$

whose extremals are invariant under the group (2).* It is then desired to specialize further these integrals by prescribing that the transversality relation† is equivalent to

$$\bar{y}' = P(y') \tag{4}$$

where $P(y')$ is a given function.‡

In the last half of the paper, there is given a geometrical construction of the focal points for the extremals (1).

§ 1. **The types of integrals whose extremals are invariant under the group (2).** In order to apply to this problem a method which Darboux§ has developed, the most general differential equation of the second order which is invariant under (2) will be required. Two independent infinitesimal transformations of the group (2) are

$$\begin{aligned} U_1f &\equiv \frac{\partial f}{\partial x} \equiv p, \\ U_2f &\equiv x\frac{\partial f}{\partial x} + (y - \alpha)\frac{\partial f}{\partial y} \equiv xp + (y - \alpha)q, \end{aligned} \tag{5}$$

with the twice extended infinitesimal transformations

$$\begin{aligned} U_1''f &\equiv p, \\ U_2''f &\equiv xp + (y - \alpha)q - y''q'' \qquad \left(q'' \equiv \frac{\partial f}{\partial y''}\right). \end{aligned} \tag{6}$$

A necessary and sufficient condition that the solutions of the equation

$$y'' = A(x, y, y')$$

* For an analogous problem, see Guldberg, "Sur une classification des problèmes du calcul des variations," Rendiconti di Circolo Matematico di Palermo, vol. 21 (1906), pp. 66–74.

† The direction $\bar{y}'$ is said to be transversal to the direction y' with respect to a function $f(x, y, y')$ if

$$f(x, y, y') + (\bar{y}' - y')f_{y'}(x, y, y') = 0.$$

‡ Stromquist, "A second inverse problem of the calculus of variations," Annals of Mathematics, Series 2, vol. 9 (1908), pp. 57–68.

§ Darboux, Leçons sur la theorie générale des surfaces, vol. 3, p. 53.

shall be invariant under $U_i''f$ $(i = 1, 2)$ is that $U_i''f = 0$ whenever $f \equiv y'' - A(x, y, y') = 0$. One finds easily from these conditions that A must have the form

$$A = \frac{F(y')}{y - \alpha}, \tag{7}$$

where F is an arbitrary function.

Hence, *the most general differential equation of the second order which is invariant under the group* (2) *has the form*

$$(y - \alpha)y'' = F(y'). \tag{8}$$

The solutions of this equation are readily found to have the form (1), which proves that these curves constitute the most general two-parameter family invariant under the group (2).

To determine the integrals whose extremals are defined by equations (1), the value y'' given by equation (8) may be substituted in Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

when differentiated out. The result is the partial differential equation for the determination of f,

$$f_{y'y'}F(y') + y'(y - \alpha)f_{y'y} + (y - \alpha)f_{y'x} - (y - \alpha)f_y = 0.$$

If Darboux's method is now applied to the solution of this equation, one obtains the following result.

The most general function f, whose extremals are invariant under the group (2), *is*

$$\begin{aligned} f = \int_0^{y'} \frac{y' - v}{F(v)} G[(y - \alpha)\rho(v),\ x - (y - \alpha)\sigma(v)]dv \\ + \lambda_0 y' + \mu_0 + y'\frac{\partial H}{\partial y} + \frac{\partial H}{\partial x}, \end{aligned} \tag{9}$$

where F and G are arbitrary functions of their arguments, H is an arbitrary function of x and y, ρ and σ are functions defined by

$$\rho(y') = \frac{1}{\varphi[\psi(y')]}, \qquad \sigma(y') = \frac{\psi(y')}{\varphi(y')}, \qquad \varphi_u[\psi(y')] = y',$$

and λ_0 and μ_0 are particular solutions of the equation

$$G[(y - \alpha)\rho(0),\ x - (y - \alpha)\sigma(0)] + \frac{\partial \lambda}{\partial x} - \frac{\partial \mu}{\partial y} = 0.$$

§ 2. **Integrals with invariant extremals and a prescribed transversality relation.** If it is required that the integrals (3) have a transversality relation of

the form (4), it is clear that f must satisfy the equation

$$f(x, y, y') + (P - y')f_{y'}(x, y, y') = 0, \tag{10}$$

which must hold for all values of x, y, y'. The solution of this linear partial differential equation has the form*

$$f = \eta(x, y)\theta(y'), \tag{11}$$

where η is an arbitrary function and

$$\log \theta(y') = \int_0^{y'} \frac{dy'}{y' - P(y')}.$$

Two methods of procedure are now possible in order to determine the function f with extremals invariant under the group (2) and having the transversality relation (4). The value of f given by the expression (9) may be compared with that defined by (11) and one may thus restrict the arbitrary functions F, G, H. A second and far simpler method, however, is to obtain Euler's equation corresponding to the function f in expression (11), and then to subject this differential equation to the invariance of the group (2). In this way conditions on the function η may be obtained.

Euler's differential equation corresponding to the function given by (11) is found to be

$$y'' = \frac{\eta_y(\theta - y'\theta') - \eta_x\theta'}{\eta\theta''} \equiv S(x, y, y'), \tag{12}$$

where $\theta' = d\theta/dy'$, $\theta'' = d^2\theta/dy'^2$. Let us suppose that θ'' is everywhere different from zero, so that the differential equation (12) is perfectly well defined. If the differential equation (12) is invariant under (2), it can be easily argued† that the partial differential equation

$$Af \equiv p + y'q + S(x, y, y')q' = 0 \qquad \left(q' = \frac{\partial f}{\partial y'}\right) \tag{13}$$

is invariant under the first extended group

$$U_1'f \equiv p,$$

$$U_2'f \equiv xp + (y - \alpha)q,$$

which, in this case, has the same infinitesimal transformations as the original group (2).

A necessary and sufficient condition that equation (13) shall be invariant under $U_i'f$ $(i = 1, 2)$ is that there shall exist functions $\mu_i(x, y, y')$ such that

$$(AU_i')f \equiv AU_i'f - U_i'Af = \mu_i(x, y, y')Af. \tag{14}$$

* Stromquist, loc. cit.

† Lie, Vorlesungen über Differential Gleichungen, p. 364.

Using the first of these relations, one finds that

$$-\frac{q'}{\eta^2\theta''}[\eta\{(\theta - y'\theta')\eta_{yx} - \eta_{xx}\theta'\} - \eta_x\{\eta_y(\theta - y'\theta') - \eta_x\theta'\}]$$
$$= \mu\left[p + qy' + \frac{\eta_y(\theta - y'\theta') - \eta_x\theta'}{\eta\theta''}q'\right]$$

which is to be an identity in p, q, q', and hence $\mu \equiv 0$. The function η must then satisfy the equation

$$(15) \qquad (\eta\eta_{yx} - \eta_x\eta_y)(\theta - y'\theta') - (\eta\eta_{xx} - \eta_x^2)\theta' = 0.$$

Remembering that $\theta'' \neq 0$ and hence that $\theta' \neq$ const., and keeping in mind, moreover, that x, y and y' are independent variables in this last equation, one finds that $(\theta - y'\theta')/\theta' =$ const. different from zero, or else the two equations $\eta\eta_{xx} - \eta_x^2 = 0$ and $\eta\eta_{yx} - \eta_x\eta_y = 0$ hold simultaneously.

In the latter case, one has

$$\frac{\partial}{\partial x}\left(\frac{\eta_y}{\eta}\right) = 0, \qquad \frac{\partial}{\partial x}\left(\frac{\eta_x}{\eta}\right) = 0,$$

from which it is easily found that η must have the form

$$\eta = me^{kx}Y(y),$$

where m and k are arbitrary constants and Y is an arbitrary function of y.

The partial differential equation (13) now reduces to

$$Af \equiv p + y'q + \frac{Y_y(\theta - y'\theta') - kY\theta'}{Y\theta''}q' = 0.$$

Making use now of the second of the relations (14), one finds after a little computation that Y must satisfy the equation

$$\frac{\theta - y'\theta'}{\theta'}\left\{\frac{Y_y}{Y} + (y - \alpha)\frac{YY_{yy} - Y_y^2}{Y^2}\right\} = k.$$

Since in the case under consideration $(\theta' - y'\theta')/\theta' =$ const. has been excluded, the equations $k = 0$ and

$$\frac{d}{dy}\left[(y - \alpha)\frac{d}{dy}\log Y\right] = 0$$

must hold true, from which it follows that

$$Y = \gamma(y - \alpha)^{\beta},$$

where β and γ are arbitrary constants of integration.

Hence, *all the integrals* (3), *whose extremals are invariant under the group* (2) *and which have the prescribed transversality relation* (4), *are of the form*

$$I = \gamma \int_{x_1}^{x_2} (y - \alpha)^\beta \theta(y') dx, \tag{16}$$

where β *and* γ *are arbitrary constants and*

$$\log \theta(y') = \int_0^{y'} \frac{dy'}{y' - P(y')}.$$

On recalling now that

$$\theta - y'\theta' = c\theta' \qquad (c = \text{const.})$$

also satisfies equation (15), one readily finds that

$$\theta = (y' + c)d,$$

where d is a constant. This is an expression linear in y' and hence $\theta'' \equiv 0$. It can be easily shown in this case that it is not possible to have a two-parameter family of extremals, and this is not in accordance with the original hypothesis. There still remains the possibility that θ'' may vanish at some isolated points of the extremals, but this is of rare occurrence in the ordinary problems of the calculus of variations and will not be considered here.

As an application of the foregoing results, it may be interesting to remark that when $P = -1/y'$, that is, when the extremals and transversals are perpendicular to each other, $\theta(y')$ takes the form

$$\theta(y') = \sqrt{1 + y'^2},$$

and, hence, the integrals (16) are

$$I = \gamma \int_{x_1}^{x_2} (y - \alpha)^\beta \sqrt{1 + y'^2}.$$

This case includes some of the commonest problems of the calculus of variations. A few examples are the problem of finding the shortest distance from a point to a curve, the problem of finding the curve joining a point and a curve along which the integral $\int_{x_1}^{x_2} \sqrt{1 + y'^2} dx/y$ shall have the least value, and all the problems mentioned at the beginning of this paper considered with one end point variable, except Newton's problem.

§ 3. **A geometrical construction of focal points for extremals invariant under the above group.** It is well known in the calculus of variations that if a particular extremal is imbedded in a one-parameter family of extremals

$$E_t: \quad y = e(x, t),$$

every one of which is cut transversally by a curve

$$G: \quad x = x(t), \qquad y = y(t),$$

then the focal point of G on a particular arc E_0 of the family is the point of contact of E_0 with the envelope of the family E_t. It is proposed to give, in this section, a construction of the focal points which holds for all families of extremals of the type

$$y = a\varphi\left(\frac{x-b}{a}\right) + \alpha \equiv a\varphi(u) + \alpha, \tag{17}$$

invariant under the group (2) for problems which have the transversality relation

$$\frac{y_t}{x_t} = -\frac{1}{Q(p)}, \tag{18}$$

where now p is used instead of y' to denote the slope along an extremal at the point (x, y), and Q represents the slope of the normal to the curve G at its point of intersection with the extremal. It is interesting to note in this connection that Miss M. E. Sinclair has given a geometrical construction of the focal points when the extremals are catenaries and the condition of transversality is orthogonality.*

In order to obtain a valid equation for the determination of the focal points, it will be convenient, in the first place, to exclude from this discussion all those transversal curves which have a horizontal tangent at the point P_1 where the extremal and transversal curves intersect. Secondly, Q must be a monotonic function of p in the neighborhood of this point of intersection, i. e., $Q'[\varphi_u(u)]$ must not vanish at $u = u_1$ (u_1 is the value of the variable u corresponding to P_1) nor in its neighborhood.

Consider, now, the one-parameter family of curves obtained from (17) whose individual curves intersect G and satisfy the relation (18) with G at the point of intersection. Then, after some computation, the contact points of those curves with their envelope are found to be determined by the equation†

$$\frac{\omega(u) - \Omega_1}{\omega(u_1) - \Omega_1} = \frac{r_1\varphi_u(u_1)}{r_1\varphi_u(u_1) + \epsilon S_1}, \tag{19}$$

where

$$\omega(u) = \frac{\varphi(u)}{\varphi_u(u)} - u,$$

$$\Omega_1 = -[u_1 + \varphi(u_1)Q(p_1)],$$

* Miss M. E. Sinclair, "On the minimum surface of revolution in the case of one variable end-point," *Annals of Mathematics*, series 2, vol. 8 (1907), p. 177.

† Bolza, Lectures on the Calculus of Variation, p. 113.

$$S_1 = \rho_1 Q_1' \left[\frac{1 + \varphi_u^2(u_1)}{1 + Q_1^2} \right]^{3/2}, \qquad Q_1 = Q(p_1),$$

ρ_1 is the radius of curvature of G, r_1 the radius of curvature of E at the point P_1, and $\epsilon = y_t/|y_t|$.

It is now desired to obtain a geometrical interpretation of equation (19). For the sake of concreteness, let it be supposed that the parametric representation of the curve G is of such a nature that as t increases, y decreases in the neighborhood of the point P_1. Then $y_t < 0$ and $\epsilon = -1$. Moreover, in accordance with the usual convention,* the positive directions of the tangent and normal to G at the point P_1 will be like those indicated in Fig. 1. It will be understood in the following drawings that the primed letters refer to the curve G while the unprimed letters to the extremal E.

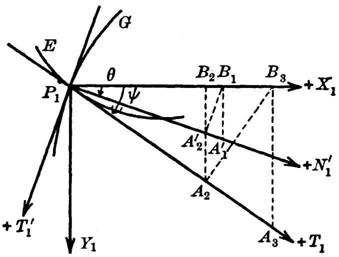

FIG. 1.

In Fig. 1

$$\theta = \tan^{-1} Q_1, \qquad \psi = \tan^{-1} \varphi_u(u_1),$$

$$P_1 A_1' = \rho_1 Q_1', \qquad P_1 B_2 = P_1 A_1' \cos^3 \theta = \frac{\rho_1 Q_1'}{(1 + Q_1^2)^{3/2}},$$

$$P_1 A_3 = P_1 B_2 \sec^3 \psi = \rho_1 Q_1' \left[\frac{1 + \varphi_u^2(u_1)}{1 + Q_1^2} \right]^{3/2} = S_1.$$

The construction of the quantity S_1 is then as follows:

On the normal to the curve G at the point P_1, lay off $P_1 A_1' = \rho_1 Q_1'$† taking care of signs. Project $P_1 A_1'$ upon $P_1 X_1$, a line through P_1 parallel to the X-axis, and so determine B_1. Project $B_1 P_1$ back again upon the normal and through A_2', the terminal point of the last projection, draw a perpendicular to $P_1 X_1$. Call A_2 the intersection of this perpendicular with the tangent to the extremal at the point P_1. The intersection of a perpendicular to the tangent at A_2 with $P_1 X_1$ will determine the point B_3, while the inter-

* See Scheffers, Anwendungen der Differential und Integralrechnung auf Geometrie, vol. 1, p. 37.

† This quantity can be constructed geometrically if the curve $Q(p)$ is supposed drawn.

section of a perpendicular to P_1X_1 at B_3 with the tangent will, in turn, give A_3. $P_1A_3 = S_1$ is the required quantity.

Having now obtained S_1, one may easily complete the construction of the focal points. It is to be understood, of course, that while the drawing in Fig. 2 is made only for a special case, the construction as given below is perfectly general.

In Fig. 2,

$$P_1C_1 = r_1, \qquad P_1R_1 = -r_1\varphi_u(u_1), \qquad P_1A_3 = \epsilon S_1,$$

$$P_1L_1 // P_1T_1', \qquad TL_1 // P_1T_1.$$

$$\begin{aligned} a[\omega(u) - \Omega_1] &= \frac{y-\alpha}{\varphi_u(u)} + (x_1 - x) + (y_1 - \alpha)Q_1 \\ &= TF + FF_1 + F_1T_1' = TT_1', \\ a[\omega(u_1) - \Omega_1] &= \frac{y_1-\alpha}{\varphi_u(u_1)} + (y_1 - \alpha)Q_1 = T_1F_1 + F_1T_1' = T_1T_1'. \end{aligned} \tag{20}$$

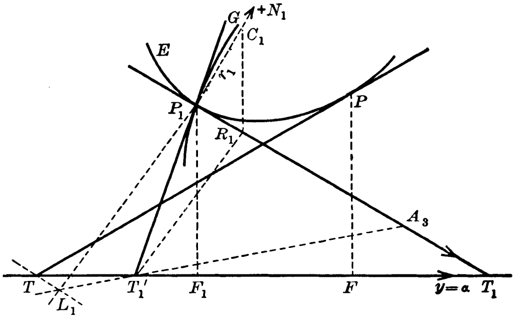

FIG. 2.

If one substitutes the expressions given by (20) in equation (19) he obtains

$$\frac{TT_1'}{T_1T_1'} = \frac{R_1P_1}{R_1P_1 + P_1A_3} = \frac{R_1P_1}{R_1A_3},$$

which, as can be seen from Fig. 2, is a true relation. Hence, the construction may be made as follows:

Draw the tangent and normal to the extremal E at the point P_1 and give these lines positive directions in accordance with the usual conventions. On the normal, lay off with proper sign $P_1C_1 = r_1$, and on the tangent make $P_1A_3 = \epsilon S_1$. Through C_1 draw a perpendicular to $y = \alpha$ and call its intersection with P_1T_1, R_1. Construct the tangent to G at the point P_1 and call T_1' its intersection with $y = \alpha$. Connect T_1' with both A_3 and R_1. Through the intersection L_1 of $T_1'A_3$ and a line P_1L_1 parallel to R_1T_1' draw TL_1 parallel to P_1T_1. The line TP tangent to E determines the focal point P.

The case $\rho_1 = 0$ possesses particular interest since then $S_1 = 0$, A_3 coincides with P_1, T coincides with T_1, and the construction reduces to the Lindelöf construction* for the point conjugate to P_1. The equation for the determination of the focal point is easily found from equation (19) to be $\omega(u) - \omega(u_1) = 0$. There are similar simplifications when the transversal curve is a straight line, i. e., when $\rho_1 = \infty$ and also when ρ_1 satisfies the equation $r_1\varphi_u(u_1) + \epsilon S_1 = 0$.

As has already been remarked, the case when the extremals and their transversals are orthogonal is of very frequent occurrence in the calculus of variations, and, hence the construction of the focal in this case deserves particular mention. Since now $Q_1 = p_1$, S_1 reduces to ρ_1, and the equation for the determination of the focal points becomes

$$\frac{\omega(u) - \Omega_1}{\omega(u_1) - \Omega_1} = \frac{r_1\varphi_u(u_1)}{r_1\varphi_u(u_1) + \epsilon\rho_1},$$

where, now

$$\Omega_1 \equiv - [u_1 + p_1\varphi(u_1)].$$

It can be easily verified that the construction (Fig. 2) is as follows:

Draw the tangent and normal to E at the point P_1. Lay off with proper sign $P_1C_1 = r_1$ on the normal and $P_1A_3 = \rho_1$ on the tangent. Construct C_1R_1 perpendicular to $y = \alpha$ and call its intersection with the tangent P_1T_1, R_1. From the point of intersection T_1' of the normal and $y = \alpha$ draw lines to A_3 and R_1. Through the intersection L_1 of $T_1'A_3$ and a line through P_1 parallel to R_1T_1' draw TL_1 parallel to P_1T_1. The line TP tangent to E determines the focal point P.

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* Lindelöf, Mathematische Annalen, vol. 2 (1870), p. 170.